(ii) **Performance**

By construction, a Bayes estimator minimizes the posterior expected loss and, hence, the Bayes risk. Often, however, we are interested in its performance, and perhaps optimality under other measures. For example, we might examine its mean squared error (or, more generally, its risk function) in looking for admissible or minimax estimators. We also might examine Bayesian measures using other priors, in an investigation of Bayesian robustness.

These latter considerations tend to lead us to look for either manageable expressions for or accurate approximations to the integrals in (3.5). On the other hand, the considerations in (i) are more numerical (or computational) in nature, leading us to algorithms that ease the computational burden. However, even this path can involve statistical considerations, and often gives us insight into the performance of our estimators.

A simplification of (3.5) is possible when dealing with independent prior distributions. If $X_i \sim f(x|\theta_i)$, $i = 1, \cdots, n$, are independent, and the prior is $\pi(\theta_1, \cdots, \theta_n) = \prod_i \pi(\theta_i)$, then the posterior mean of θ_i satisfies

$$E(\theta_i|x_1, \ldots, x_n) = E(\theta_i|x_i), \tag{3.6}$$

that is, the Bayes estimator of θ_i only depends on the data through x_i. Although the simplification provided by (3.6) may prove useful, at this level of generality it is impossible to go further.

However, for exponential families, evaluation of (3.5) is sometimes possible through alternate representations of Bayes estimators. Suppose the distribution of $\mathbf{X} = (X_1, \ldots, X_n)$ is given by the multiparameter exponential family (see (1.5.2)), that is,

$$p_{\boldsymbol{\eta}}(\mathbf{x}) = \exp\left\{\sum_{i=1}^{s} \eta_i T_i(\mathbf{x}) - A(\boldsymbol{\eta})\right\} h(\mathbf{x}). \tag{3.7}$$

Then, we can express the Bayes estimator as a function of partial derivatives with respect to $\mathbf{x}$. The following theorem presents a general formula for the needed posterior expectation.

Theorem 3.2 *If* $\mathbf{X}$ *has density (3.7), and* $\boldsymbol{\eta}$ *has prior density* $\pi(\boldsymbol{\eta})$*, then for* $j = 1, \ldots, n$,

$$E\left(\sum_{i=1}^{s} \eta_i \frac{\partial T_i(\mathbf{x})}{\partial x_j} |\mathbf{x}\right) = \frac{\partial}{\partial x_j} \log m(\mathbf{x}) - \frac{\partial}{\partial x_j} \log h(\mathbf{x}), \tag{3.8}$$

where $m(\mathbf{x}) = \int p_{\boldsymbol{\eta}}(\mathbf{x})\pi(\boldsymbol{\eta})\,d\boldsymbol{\eta}$ *is the marginal distribution of* X*. Alternatively, the posterior expectation can be expressed in matrix form as*

$$E(\mathcal{T}\boldsymbol{\eta}) = \nabla \log m(\mathbf{x}) - \nabla \log h(\mathbf{x}), \tag{3.9}$$

where $\mathcal{T} = \{\partial T_i/\partial x_j\}$.

Proof. Noting that $\partial \exp\{\sum \eta_i T_i\}/\partial x_j = \sum_i \eta_i(\partial T_i/\partial x_j) \exp\{\sum \eta_i T_i\}$, we can write

$$E\left(\sum \eta_i \frac{\partial T_i(\mathbf{x})}{\partial x_j} |\mathbf{x}\right) = \frac{1}{m(\mathbf{x})} \int \sum_i \left[\eta_i \frac{\partial T_i}{\partial x_j}\right] e^{\Sigma \eta_i T_i - A(\boldsymbol{\eta})} h(\mathbf{x})\pi(\boldsymbol{\eta})\,d\boldsymbol{\eta}$$