map of ringed spaces $\operatorname{Spec}\mathbb{k}(x) \to \operatorname{Spec}\mathbb{k}[x]_{(x)}$ *sending the point of* $\operatorname{Spec}\mathbb{k}(x)$ *to* $[(x)]$*, and the pullback map* $f^{\#}\mathcal{O}_{\operatorname{Spec}\mathbb{k}(x)} \to \mathcal{O}_{\operatorname{Spec}\mathbb{k}[x]_{(x)}}$ *is induced by* $\mathbb{k}[x]_{(x)} \hookrightarrow \mathbb{k}(x)$*. Show that this map of ringed spaces is not of the form described in Exercise 4.7.4.*

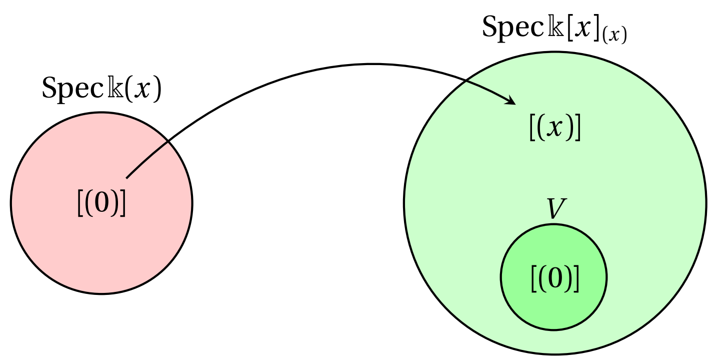

Figure 4.12: Exercise 4.15.2

PROOF. $\operatorname{Spec}\mathbb{k}(x)$ is one-point space while $\operatorname{Spec}\mathbb{k}[x]_{(x)}$ has two points. Their open subsets has been depicted in Figure 4.12 as circles. Let $(f, f^{\#})$ be a morphism such that $[(0)] \mapsto [(x)]$. Then $f^{\#}$ must be a morphism $\mathcal{O}_{\operatorname{Spec}\mathbb{k}[x]_{(x)}} \to f_*\mathcal{O}_{\operatorname{Spec}\mathbb{k}(x)}$. We define

$$f^{\#}(\operatorname{Spec}\mathbb{k}[x]_{(x)}) : \mathcal{O}_{\operatorname{Spec}\mathbb{k}[x]_{(x)}}(\operatorname{Spec}\mathbb{k}[x]_{(x)}) \to f_*\mathcal{O}_{\operatorname{Spec}\mathbb{k}(x)}(\operatorname{Spec}\mathbb{k}[x]_{(x)})$$

to be

$$\mathbb{k}[x]_{(x)} \to \mathbb{k}(x);$$

and define

$$f^{\#}(V) : \mathcal{O}_{\operatorname{Spec}\mathbb{k}[x]_{(x)}}(V) \to f_*\mathcal{O}_{\operatorname{Spec}\mathbb{k}(x)}(V) = \{0\}$$

and

$$f^{\#}(\varnothing) : \mathcal{O}_{\operatorname{Spec}\mathbb{k}[x]_{(x)}}(\varnothing) \to f_*\mathcal{O}_{\operatorname{Spec}\mathbb{k}(x)}(\varnothing) = \{0\}$$

to be zero morphisms. $f^{\#}$ is clearly a morphism an $(f, f^{\#})$ is a morphism of ringed spaces. If $(f, f^{\#})$ is a morphism of the form described in Exercise 4.7.4, then $f^{\#}$ must be deduced form $\mathbb{k}[x]_{(x)} \to \mathbb{k}(x)$. Therefore $[(0)]$ must be mapped to $[(0)]$, a contradiction. ∎