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Contents

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THE INVERSE 1-MEDIAN PROBLEM ON A PLANE AND ON A CYCLE WITH NEGATIVE WEIGHTS

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Abstract

This article considers two problems, the first one is the inverse Fermat-Weber problem, provided the Euclidean 1-median is a vertex and the second one is the inverse 1-maxian problem on cycles. For any two problems, the aim is to change the vertex weights at minimum total cost with respect to given modification bounds such that a prespecified vertex becomes 1-median. If the prespecified point coincides with one of the given n points in the plane, it is shown that the corresponding inverse problem can be written as convex problem and hence is solvable in polynomial time to any fixed precision. We show that the inverse 1-maxian problem on a cycles with positive edge-lengths and unit cost can be solved in $O(n^2)$-time.

1. Introduction

In recent years inverse optimization problems found an increased interest. The *inverse optimization problem* consists in changing parameters of the problem at minimum cost such that a prespecified solution

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becomes optimal. Recently, inverse p-median problem has been investigated by Burkard, Pleschiutsching and Zhang [2]. They showed that the discrete inverse p-median problem with real weights can be solved in polynomial time provided p is fixed and not an input parameter. They presented a greedy-like $O(n \log n)$-time algorithm for the inverse 1-median problem in the plane provided the distances between the points are measurde in the Manhattan or maximum metric. Also, they showed that the inverse 1-median problem on a cycle with positive vertex weights can be solved in $O(n^2)$ time. The inverse Fermat-Weber problem was studied by Burkard, Galavii, and Gassner [1]. The authors derived a combinatorial approach which solves the problem in $O(n \log n)$-time for unit cost and under the assumption that the prespecified point that should become a 1-median does not coincide with a given point in the plane. Galavii [4] showed that the inverse 1-median problem on a tree with positive weights can be solved in linear time. In this paper we investigate the inverse Fermat-Weber problem on a plane, provided the Euclidean 1-median is a vertex and also the inverse 1-median problem on cycles with negative weights.

2. The inverse Fermat-Weber problem on a plane with vertex 1-median

Given n points $P_1, P_2, ..., P_n$ in a metric space (X, d) and positive weights $w_1, w_2, ..., w_n$ the 1-*median problem* asks for a point $P \in X$ which minimizes

$$\sum_{i=1}^{n} w_i d(P_i, P).$$

In the *inverse* 1-*median problem* a point P_0 is given in addition to the points $P_1, P_2, ..., P_n$. The weight of these points have to be modified within given bounds $[\underline{w}_i, \overline{w}_i]$ such that P_0 becomes a 1-median and the sum of weight changes [or: the cost for the weight changes] is as small as possible.

Definition 2.1. If $P_0 \neq P_i$ for all $i = 1, 2, ..., n$ the resultant force $R(P_0)$ at P_0 given by:

$$R(P_0) := \sum_{i=1}^{n} \frac{w_i}{d(P_i, P_0)}(P_i - P_0);$$

and for $P_i = P_j$ for some $j = 1, 2, ..., n$,

$$R(P_0) := \max(\|R_j\| - w_j, 0)\frac{R_j}{\|R_j\|\|}.$$

Differential calculus tells that P_0 is a 1-median if and only if the resultant $R(P_0) = 0$. In case that $P_i \neq P_j$ we can assume that P_0 lies in the interior of the convex hull of the points P_i, $i = 1, 2, ..., n$ and that P_0 is the origin. Next we project the given points on the unit circle. The point P_0=(0,0) is a Euclidean 1-median if and only if

$$R_x(w) := \sum_{i=1}^{n} w_i x_i = 0,$$

$$R_y(w) := \sum_{i=1}^{n} w_i y_i = 0.$$

Since the Euclidean distance is invariant with respect to rotation and reflection, we can always assume that

$$R_x(w) = 0,$$

$$R_y(w) \leq 0.$$

If $R_y(w) = 0$, then the weights w_i, $i = 1, 2, ..., n$, provide an optimal solution. By this assumptions, in the case that $P_i \neq P_j$, it is shown that by Burkard, Galavii and Gassneer [1] that the inverse Fermat-Weber problem can be solved in $O(n \log n)$ time.

Now we consider the case that the prespecified point coincides with one of the given points. We have the following result.

Theorem 2.2. *P_0 is the optimal location if and only if*

$$w_0^2 \geq \left(\sum_{i=1}^{n} w_i \frac{x_i - x_0}{d(P_i, P_0)} \right)^2 + \left(\sum_{i=1}^{n} w_i \frac{y_i - y_0}{d(P_i, P_0)} \right)^2$$

Thus the above Theorem yields point $P_0 = (0, 0)$ is 1-median if and only if

$$R_x^2(w) + R_y^2(w) \leq w_0^2$$

holds. This condition does not lead to a convex problem. However, it is possible to fix the optimal weight in advance

Lemma 2.3. *There exists an optimal solution w^* with*

$$w_0^* = \min\{\overline{w}_0, \sqrt{R_x^2(w) + R_y^2(w)}\}.$$

The above Lemma implies that the weight of P_0 can be fixed. After modifying the weight of P_0 the remaining problem is convex and can be solved by any algorithm for convex programming.

Theorem 2.4. *If the prespecified point is one of the given n points, then an optimal solution (to any fixed precision) of the inverse Fermat-Weber problem with unit cost can be computed in polynomial time.*

3. Inverse 1-median problem on a cycle with negative vertex weights

For the classical median problem (the vertex weights are positive)on a network, it has been shown by Hakimi [5] that there always exists a node which is optimal. That is not the case for the maxian problem. Church and Garfinkel [3] have studied the 1-maxian problem on a network. The importance of that paper is that it specifies a finite set of points containing an optimal solution to the maxian problem on a network. In the 1-maxian problem we want to locate one facility such that the sum of positive weighted distances of clients to the facility is maximized. The objective of the 1-maxian problem is to find a point x for which

$$f(x) = \sum_{i=0}^{n} w_i d(x, i) \tag{3.1}$$

is maximum. As has been shown by Church and Garfinkel [3], there exists a point x^* which maximizes (3.1) such that $x^* \in B$ where B is the set of bottleneck point of network. Using these facts, the inverse 1-maxian problem on a cycle can be rewrite as a linear program that has been analyzed by Burkard, Pleschiutschnig, and Zhang [2]. Thus, we have the following Proposition.

Proposition 3.1. *The inverse* 1*-maxian problem on a cycle with positive edge-lengths and unit cost can be solved in* $O(n^2)$*-time.*

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INVERSE UNDESIRABLE CENTER LOCATION OPTIMIZATION ON GRAPHS

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Abstract

In this work, an inverse undesirable center location problem is considered in which our goal is to modify the edge lengths of the underlying graph within given bounds at the minimum total cost such that a predetermined point of the graph becomes an undesirable center location under the new edge lengths. The cost is proportional to the increase or decrease, resp., of the edge length. The total cost is defined as sum of all cost incurred by length modifications . A novel combinatorial algorithm with linear time complexity is developed for obtaining an optimal solution of this inverse location model.

1. Introduction

Undesirable facility location problems are basic models in location theory in which customers no longer consider the facilities desirable, but attempt to have them as far away as possible from their own locations. Examples of such facilities include nuclear reactors, military installations, stadiums. Two well-known models in undesirable location optimization are the *undesirable center* and the *undesirable median* problems.

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Whereas in the undesirable center problem the task is to find the best location of one or more facilities such that the minimum (weighted) distance between customers and the closest facility is maximized (see e.g. [4], [5]), the goal for an inverse undesirable location problem is to modify specific parameters (like edge lengths) of a given undesirable location problem in the cheapest possible way subject to certain modification bounds such that one or more prespecified locations become optimal under the new parameter values. To the best of our knowledge, inverse undesirable center location problems have not been investigated until now. Within the context of desirable models, however, the NP -hardness of the inverse center location problem with edge length modification on directed graphs was proved by [3]. Later, Alizadeh et al. [1], [2] developed exact algorithms for variants of the inverse absolute and vertex center location problems on trees.

In this paper, we consider the inverse undesirable center location problem with edge length modification on graphs and propose a new linear algorithm which is based on a modified binary search manner.

2. The undesirable center location problem and its inverse model

Let $G = (V(G), E(G))$ be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. Every edge $e \in E(G)$ has a positive length $\ell(e)$. Let $d_\ell(u, v)$ denote the shortest path distance between two vertices u and v under the edge lengths ℓ. It is said that point p lies in graph G, $p \in G$, if p coincides with a vertex or lies on an edge $e = uv$ with endpoints $u, v \in V(G)$. The *undesirable center location problem* on G asks for an optimal solution to

$$\begin{aligned} \text{maximize} \quad & \min_{v \in V(G)} d_\ell(v, p) \\ \text{subject to} \quad & p \in G. \end{aligned} \tag{2.1}$$

An optimal solution p^* of problem (2.1) is called an *undesirable center location* on the given graph G. The undesirable centers of G can be obtained in $\mathcal{O}(|E(G)|)$-time according to the following basic lemma.

Lemma 2.1. *(optimality criterion)*
For an unweighted graph G, the midpoint of a diameter edge is an undesirable center location.

In contrast to the classical undesirable center problem (2.1), the inverse undesirable center location problem on a graph is stated as follows: Let a graph $G = (V(G), E(G))$ with positive edge lengths $\ell(e)$, $e \in E(G)$, be given. Let s be a prespecified *interior point* (i.e.,

$s \notin V(G)$) on a specific edge e^s of G which divides e^s into two edge-segments e_1^s and e_2^s satisfying

$$\ell(e_1^s) + \ell(e_2^s) = \ell(e^s) \quad , \quad e_1^s \cap e_2^s = \{s\} \quad , \quad \ell(e_2^s) \leq \ell(e_1^s).$$

We want to modify the edge (and edge-segment) lengths in the cheapest possible way such that the prespecified point s becomes an undesirable center location under the modified edge lengths. Let $\hat{E} = \{e_1^s, e_2^s\} \cup E(G)\backslash\{e^s\}$. Suppose that we incur the nonnegative cost $c^+(e)$ if $\ell(e)$, $e \in \hat{E}$, is increased by one unit and we incur the nonnegative cost $c^-(e)$ if $\ell(e)$ is reduced by one unit. Moreover, we are not allowed to modify the lengths arbitrarily. Therefore, let $u^+(e)$ and $u^-(e)$ be the maximum permissible amounts by which length $\ell(e), e \in \hat{E}$, can be increased and reduced, respectively. We can now state the *inverse undesirable center location problem* (IOCP for short) on G as follows:

Modify the lengths $\ell(e)$, $e \in E \cup \{e_1^s, e_2^s\}$, to $\tilde{\ell}(e)$ such that the following three statements (i), (ii) and (iii) are satisfied:

(i) The prespecified point s becomes an undesirable center location on graph G with respect to new lengths $\tilde{\ell}$.

(ii) The cost function

$$\sum_{e \in \hat{E}} \Big(c^+(e) \max\{0, \tilde{\ell}(e) - \ell(e)\} + c^-(e) \max\{0, \ell(e) - \tilde{\ell}(e)\} \Big)$$

for changing the edge lengths on G is minimized.

(iii) The new edge (or edge-segment) lengths lie within the given modification bounds

$$-u^-(e) \leq \tilde{\ell}(e) - \ell(e) \leq u^+(e) \qquad \text{for all } e \in \hat{E}.$$

We are now going to present briefly our solution method.

3. Optimal solution approach

According to Lemma 2.1, our *generic solution idea* for solving IOCP is as follows: Either increase or reduce the lengths $\ell(e)$, $e \in \hat{E}$, at minimum total cost subject to the given modification bounds $u^-(e)$ and $u^+(e)$ such that the equalities

$$\tilde{\ell}(e^s) = \max\{\tilde{\ell}(e) : e \in \hat{E}\} \quad , \quad \tilde{\ell}(e_1^s) = \tilde{\ell}(e_2^s)$$

are satisfied. Note that an optimal modification of IOCP may either reduce or increase the length $\ell(e_1^s)$. Hence we have to take into consideration both of these cases.

Case 1. The length $\ell(e_1^s)$ is increased in an optimal solution of IOCP.

In this case, both lengths $\ell(e_1^s)$ and $\ell(e_2^s)$ are to be increased and the other lengths $\ell(e)$, $e \neq e^s$, may be reduced. We have shown that in the current case the solution of IOCP is reduced to the solution of the nonlinear programming problem

$$\begin{aligned} \min \quad & f(z) = \left(\sum_{i=1}^{2} c^+(e_i^s)\right)(\frac{1}{2}z - \ell(e_1^s)) + \sum_{e:\ \ell(e)\geq z} c^-(e)\,(\ell(e) - z) \\ \text{s.t.} \quad & \frac{1}{2}z - \ell(e_1^s) \leq \min\{u^+(e_1^s), u^+(e_2^s)\}, \\ & \ell(e) - z \leq u^-(e) \qquad \text{for all } e \in \hat{E} \text{ with } \ell(e) \geq z, \\ & 2\ell(e_1^s) \leq z \leq \max\{2\ell(e_1^s)\ ,\ \ell(e)\ ; e \in \hat{E}\}, \end{aligned} \tag{3.1}$$

where we have $\frac{1}{2}z = \tilde{\ell}(e_1^s)$. We have constructed a new procedure which solve the problem (3.1) in $\mathcal{O}(|E(G)|)$ time.

Case 2. The length $\ell(e_1^s)$ is reduced in an optimal solution of IOCP.

In this case, length $\ell(e_2^s)$ may be increased and the lengths $\ell(e_1^s)$ as well as $\ell(e), e \neq e^s$ may be reduced. In an analogous way, the solution of IOCLP is also reduced to the optimal solution of a specific nonlinear program which can be solved in $\mathcal{O}(|E(G)|)$ time.

From the optimal solution of the mentioned nonlinear programs with smaller objective value, an optimal solution for IOCP is derived.

Altogether, we get

Theorem 3.1. *The inverse undesirable center location problem can be solved in $\mathcal{O}(|E(G)|)$-time on a graph.*

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A QUADRATIC L-SHAPED ALGORITHM FOR TWO-STAGE STOCHASTIC LINEAR PROGRAMMING (SLP)

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Abstract

Stochastic programming is a technique for optimization in the presence of uncertainty which typically leads to very large problem sizes. Here, we present a modified version of the L-shaped method and reduce linear master and linear recourse programs to unconstrained maximization of concave differentiable piecewise quadratic functions.

1. Introduction

In mathematical linear programming, matrix of coefficients and vectors are exact values. However, in practice, the problem data are not definite because of many reasons like error in measurement, incomplete information about future and events which have not occurred yet. In stochastic programming, some data are random variables with a specific possibility distribution. Before presenting the mathematical formulation of the two-stage stochastic linear program (SLP) model, we introduce some notation. Let $(\Omega; \vartheta; P)$ be a discrete probability space and consider $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ as the set of scenarios with

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associated probabilities $\{\rho_1, \rho_2, \ldots, \rho_N\}$ such that $\sum_{i=1}^{N} \rho_i = 1$. In this paper, consider the following two-stage stochastic linear program (SLP) with fixed recourse and a finite number of scenarios [2]:

$$\min_{x \in X} \quad f(x) = c^T x + \phi(x), \quad X = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}, \tag{1.1}$$

where

$$\phi(x) = E(Q(x, \omega)) = \sum_{i=1}^{N} Q(x, \omega_i)\rho_i,$$

and

$$Q(x, \omega) = \min_{y \in \mathbb{R}^{n_2}} \{q(\omega)^T y \mid Wy = h(\omega) - T(\omega)x \ , \ y \geq 0\}. \tag{1.2}$$

Here E represents the expectation with respect to $\omega \in \Omega$. In the second stage $q(.) \in \mathbb{R}^{n_2}$, $h(.) \in \mathbb{R}^{m_2}$ and matrix $T(.) \in \mathbb{R}^{m_2 \times n}$ for each realization ω and $W \in \mathbb{R}^{m_2 \times n_2}$ is the recourse matrix which we are taking here as fixed. Also, in the first stage, $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. In this paper, matrices A and W are assumed to have full row rank and $m << n$ and $m_2 << n_2$.

The stochastic program (1.1)-(1.2) can be reformulated [2] as the following deterministic equivalent program:

$$\begin{aligned} \min \quad & c^T x + \sum_{i=1}^{N} \hat{q}_i y_i \\ s.t. \quad & Ax = b, \\ & T_i x + W y_i = h_i, \ i = 1, \ldots, N, \\ & x \geq 0, \ y_i \geq 0, \ i = 1, \ldots, N, \end{aligned} \tag{1.3}$$

where $T_i := T(\omega_i), h_i := h(\omega_i), y_i := y(\omega_i), q_i := q(\omega_i)$ and $\hat{q}_i := \rho_i q_i$ for each realization ω_i of the random variable ω. Usually N is a very large number. Hence, the stochastic linear programs (1.1)-(1.2) or (1.3) can become huge and very difficult to solve. The challenge of solving such problems has led to many interesting computational and theoretical developments and has provided a motivation for more study. L-shaped decomposition methods split the original problem into a master problem (1.1) and a series of independent subproblems (1.2) for each $\omega \in \Omega$. To solve this problem, the LP method has been used for solving both primal and dual subproblems and master problem in the L-shaped method. In this paper, we introduce a new method for SLP(1.1)-(1.2). In this method, we apply the fast algorithm of an augmented Lagrangian method into the L-shaped method to improve the speed in solving two-stage stochastic linear program (1.1) with respect to traditional methods [3].

2. L-shaped decomposition

The L-shaped decomposition method consists of three main steps: generating feasibility cuts, optimality cuts and solving the master problem. In this algorithm, two types of constraints are sequentially added to linear master problem: feasibility cuts and optimality cuts. To solve this problem, the LP method has been used for both master and subproblems. In [1, 2, 4], the algorithm of linear L-shaped method can be seen. In this paper, we introduce quadratic L-shaped method in which unconstrained quadratic program is solved to generate feasibility and optimality cuts.

3. Feasibility cut

Feasibility cut tests whether the recourse problem is feasible for current vector x^ν for all $i = 1, \ldots, N$ or not. If not, this means that for some i, there is a hyperplane separating $h_i - T_i x^\nu$ and set $\{t \mid t = Wy, y \geq 0\}$. If we name the hyperplane $\{x \mid \sigma x = 0\}$, this hyperplane must satisfy $\sigma^T t \leq 0$ for all $t \in \{t \mid t = Wy, y \geq 0\}$ and $\sigma^T(h_i - T_i x^\nu) > 0$.

In this paper, instead of Slyke and Wets method [4], we solve the following quadratic program

$$\min_{y \in \mathbb{R}^{n_2}_+} \eta' = \frac{1}{2}\|Wy - (h_i - T_i x^\nu)\|^2, \tag{3.1}$$

and set the feasibility cut $\sigma^T t \leq 0$ in which $\sigma = \frac{(h_i - T_i x^\nu) - Wy}{\|(h_i - T_i x^\nu) - Wy\|}$.

4. Optimality Cut

In this section, we present the augmented Lagrangian method for solving the recourse subproblem and generating an optimality cut.

Theorem 4.1. [3] *Consider the following maximization problem*

$$\max_{p \in \mathbb{R}^{m_2}} S(p, \beta, \hat{y}) \tag{4.1}$$

in which $\beta, \hat{y}$ are constants and

$$S(p, \beta, \hat{y}) = (h - Tx^\nu)^T p - \frac{1}{2}\|(\hat{y} + W^T p - \beta \hat{q})_+\|^2. \tag{4.2}$$

Also, assume that the solution set Y_* of (1.2) is non-empty and the rank of sub-matrix W_l of W corresponding to nonzero components of $\hat{y}_*$ (the *projection of $\hat{y}$ on Y_*) is m_2. In such a case, there is β^* which for all $\beta \geq \beta^*$, $\hat{y}_* = (\hat{y} + W^T p(\beta) - \beta\hat{q})_+$ where $p(\beta)$ is the point obtained from solving (4.1).*

5. Numerical Results

The proposed algorithm was applied to solve 3 random generated SLPs. Table (1) compares quadratic L-shaped method with linear L-shaped method. We used projection and generalized Newton methods for (3.1) and (4.1) respectively. Also, linprog function of MATLAB was used in linear L-shaped method for solving linear programs in each iteration. As a criterion of the solution accuracy, the Chebyshev norms of residual vectors were calculated:

$$\Delta_1 = \|Ax-b\|_\infty, \Delta_2 = \max_i \|T_i x+W y_i-h_i\|_\infty, \Delta_3 = |c^T x+\sum_{i=1}^{N} \hat{q}_i y_i-f^*|,$$

where f^* is the optimal value of (1.3). Also, d and d_2 are the density of matrices A and W respectively.

Table 1. Comparative between quadratic L-shaped method (QLM) and linear L-shaped method (LLM)

$\mathbf{m \times n \times d}$	$\mathbf{m_2 \times n_2 \times d_2}$	solver	$\mathbf{\Delta_1}$	$\mathbf{\Delta_2}$	$\mathbf{\Delta_3}$	time
2*50 $\times$ 500 $\times$ 0.5	2*100 $\times$ 2e3 $\times$ 0.1	**QLM**	7.5033e-11	6.1846e-11	5.0204e-10	0.506
		LLM	7.2236e-08	3.0177e-09	2.1944e-08	0.782
2*50 $\times$ 500 $\times$ 0.5	2*100 $\times$ 2e3 $\times$ 0.01	**QLM**	7.2760e-12	7.8444e-12	2.1828e-10	0.191
		LLM	8.1059e-11	1.5726e-08	6.3192e-09	0.724
2*100 $\times$ 1e3 $\times$ 0.1	2*100 $\times$ 1e3 $\times$ 0.1	**QLM**	8.2537e-11	4.8658e-11	4.9613e-10	0.687
		LLM	4.5475e-12	2.2250e-08	1.7404e-08	0.997

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