as shown in Figure 3.4. Find $\partial f/\partial x$ in terms of the partial derivatives of the function with respect to r, θ and ϕ.

Solution Using the chain rule, we have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial f}{\partial \phi}\frac{\partial \phi}{\partial x}$$

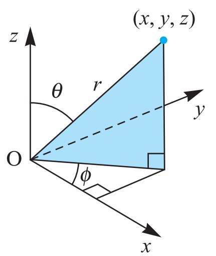

Figure 3.4 Spherical polar coordinates.

From Figure 3.4, $r^2 = x^2 + y^2 + z^2$, $\tan\phi = y/x$ and $\tan\theta = (x^2 + y^2)^{1/2}/z$, so that

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \sin\theta\cos\phi$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) = -\frac{y}{x^2 + y^2} = -\frac{\sin\phi}{r\sin\theta}$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left\{\tan^{-1}\left[\frac{(x^2 + y^2)^{1/2}}{z}\right]\right\} = \frac{xz}{(x^2 + y^2 + z^2)(x^2 + y^2)^{1/2}}$$

$$= \frac{\cos\phi\cos\theta}{r}$$

Thus

$$\frac{\partial f}{\partial x} = \sin\theta\cos\phi\frac{\partial f}{\partial r} - \frac{\sin\phi}{r\sin\theta}\frac{\partial f}{\partial \phi} + \frac{\cos\phi\cos\theta}{r}\frac{\partial f}{\partial \theta}$$

Example 3.6 The Laplace equation in two dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

where x and y are rectangular cartesian coordinates. Show that expressed in polar coordinates (r, θ), where $x = r\cos\theta$ and $y = r\sin\theta$, the Laplace equation may be written

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Solution Using the chain rule, we have

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

and

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta$$

Similarly

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$